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Limiting Solutions and Linearised Analysis of Micropolar Flow Driven by a Porous Stretching Surface

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Abstract

A perturbation analysis is used to obtain closed form solutions for self-similar boundary layer flow of a micropolar fluid driven by a porous stretching sheet. Certain limiting solutions to the governing equations are also discussed. Our analysis is compared with computed results where possible.

1 Introduction

The theory of micropolar fluids was originally formulated by Eringen [3] to model the non-Newtonian flow of fluids containing rotating micro-constituents. In essence, Eringen's theory introduces new quantities (eg. microrotation) and new constitutive equations which must be solved simultaneously with the usual equations for Newtonian flow.

Here we analyse the self-similar boundary layer flow of a micropolar fluid driven by a porous stretching surface. The potential relevance of the present problem to industrial applications such as metal drawing or polymer sheet extrusion has motivated a number of previous studies, and the most directly relevant are those of Chiam [1], Hady [4], Heruska et al. [6] and Hassanien & Gorla [5]. These studies used numerical methods to obtain flow solutions, with the exception of Hady, who obtained a closed form solution using the method of successive approximation.

In a recent paper, an alternative perturbation analysis was undertaken by Desseaux & Bellalij [2], and closed form solutions for the limiting case of very large mass transfer through the surface were obtained. However, for the case of an impermeable sheet or moderate surface mass transfer, they were unable to complete the analysis. Instead, solutions to first order were obtained numerically.

The purpose of our present work is two-fold. We revisit the perturbation analysis, and demonstrate that closed form solutions to first order in the perturbing parameter can be derived. Certain limiting solutions to the governing equations are also discussed. Our results are verified by comparison with computed solutions where possible.

2 Defining Equations

For steady laminar self-similar flow, the dimensionless boundary-layer equations for an incompressible micropolar fluid in an otherwise quiescent medium are [2]

$$F_1''' = -F_1 F_1'' + (F_1')^2 - C_1 F_2' \tag{1a}$$

$$C_2 F_2'' = F_1'' + 2F_2 \tag{1b}$$

In the above, $F_1(\eta)$ is the streamfunction, $F_2(\eta)$ is the microrotation (or angular velocity) whose direction of rotation is normal to the plane of the flow, and a prime denotes differentiation with respect to η, the normal distance from the sheet. Compared with classical Newtonian fluids, the equations include the microrotation F_2 and the dimensionless constants C_1 and C_2.

For the present study, the appropriate boundary conditions are

$$F_1'(0) = 1, \quad F_1(0) = -V_1, \quad F_2(0) = 0, \quad \lim_{\eta\to\infty} F_1'(\eta) = 0, \quad \lim_{\eta\to\infty} F_2(\eta) = 0 \tag{2}$$

where $V_1 < 0$ corresponds to suction, and $V_1 > 0$ corresponds to injection.

The problem as defined contains the parameters V_1, C_1 and C_2. For a given micro-element shape, C_1 represents a measure of the micro-element concentration, and varies in the range $0 \leq C_1 < 1$. This range suggests that C_1 can be used as a perturbing parameter, as discussed below. The parameter C_2 may be a function of the concentration, relative size and shape of the micro-constituents and must be non-negative [3]. Values near $C_2 = 2$ were chosen in earlier studies, but this parameter could vary appreciably [6].

3 Perturbation Analysis

On choosing $\epsilon = C_1$ as a perturbing parameter, we expand the similarity variables using

$$\begin{aligned} F_1 &= f_0 + \epsilon f_1 + \epsilon^2 f_2 + \ldots \\ F_2 &= g_0 + \epsilon g_1 + \epsilon^2 g_2 + \ldots \end{aligned} \tag{3}$$

and substitute them into the boundary-layer equations (1).

By collecting terms in equal powers of ϵ, a heirachy of ordinary differential equations for the functions f_n and g_n can be obtained. The first two equations for f_n are [2]

$$f_0''' + f_0 f_0'' - (f_0')^2 = 0 \tag{4a}$$

$$f_1''' + f_0 f_1'' - 2f_0' f_1' + f_0'' f_1 = -g_0' \tag{4b}$$

For $n \geq 0$, the microrotation functions g_n may be obtained via solution of

$$C_2 g_n'' - 2g_n = f_n'' \tag{5}$$

The appropriate boundary conditions for equations (4) and (5) are

$$f_0(0) = -V_1, \quad f_0'(0) = 1, \quad g_0(0) = 0, \quad \lim_{\eta\to\infty} f_0' = \lim_{\eta\to\infty} g_0 = 0 \quad \text{for } n = 0, \tag{6a}$$

$$f_n(0) = 0, \quad f_n'(0) = 0, \quad g_n(0) = 0, \quad \lim_{\eta\to\infty} f_n' = \lim_{\eta\to\infty} g_n = 0 \quad \text{for } n > 0. \tag{6b}$$

3.1 Zeroth Order Solution for F_1 and F_2

The ordinary differential equation (4a) is non-linear, third order and autonomous. Using techniques such as reduction of order, scale invariance or conversion into equidimensional form (see, eg. [7]), we found only two closed-form singular solutions to (4a), namely

$$f_0 = \beta - \alpha e^{-\beta\eta} \qquad \text{and} \qquad f_0 = 6/(\beta + \eta) \tag{7}$$

where α and β are constants.

In general, a solution for f_0 with three arbitrary constants is needed. However, for $f_0 = \beta - \alpha e^{-\beta\eta}$ the condition $\lim_{\eta\to\infty} f_0' = 0$ can be met provided that $\beta \geq 0$. With this restriction, a solution to (4a) which satisfies the relevant boundary conditions is

$$f_0 = -V_1 + \frac{1-\exp(-\beta\eta)}{\beta} = \beta(1-\omega) \tag{8a}$$

$$\text{where} \quad \beta = \frac{1}{2}(-V_1 + \sqrt{V_1^2+4}) \quad \text{and} \quad \omega = \omega(\eta) = \frac{\exp(-\beta\eta)}{\beta^2} \tag{8b}$$

To obtain the solution for g_0, we let $\gamma = \frac{1}{\beta}\sqrt{\frac{2}{C_2}}$ and substitute (8) into (5) to obtain

$$g_0 = \frac{\beta^4}{4}\eta\omega \quad \text{for } \gamma = 1, \quad \text{and} \quad g_0 = \frac{\beta^3}{2}\frac{\gamma^2}{\gamma^2-1}(\omega - \beta^{2\gamma-2}\omega^\gamma) \quad \text{for } \gamma \neq 1. \tag{9}$$

In the limit as $\gamma \to 1$, the second expression for g_0 simplifies to the first, as expected.

3.2 Other limiting solutions

The solutions described above are exact for the limiting case $C_1 = 0$, and correspond to an "uncoupled" flow where the macroscopic motion is unaffected by the microrotations.

The flow behaviour for a wide range of C_2 values is also of interest. For the limiting case of C_2 large, an expansion using $\epsilon = 1/C_2$ can be attempted. With this choice, (1b) may be written as $F_2'' = \epsilon(F_1'' + 2F_2)$, and the leading order solution to (1) is again $f_0 = \beta(1-\omega)$, but with $g_0 = 0$.

For $\epsilon = 1/C_2$, no higher order approximations can be obtained via a continuation of the perturbation analysis, as the method fails due to inconsistencies in the boundary conditions. Nevertheless, the leading order solutions suggest that Newtonian flow will result from either $C_1 \to 0$ or $C_2 \to \infty$, but with different behaviour for the microrotation.

The limiting case of C_2 small is also of interest, and an expansion using $\epsilon = C_2$ suggests itself here. With this choice, we find to leading order that $F_2 \approx -F_1''/2$, which can be expected to represent the flow behaviour except for a narrow wall region where the boundary conditions for F_1'' and F_2 will not in general match. Nevertheless, replacing F_2 by $-F_1''/2$ in (1a) yields

$$(1 - C_1/2)F_1''' + F_1F_1'' - (F_1')^2 = 0 \tag{10}$$

The solution of (10) is best obtained by rescaling $F_1 = \sqrt{1-C_1/2}\ \hat{F}_1$ and $\eta = \sqrt{1-C_1/2}\ \hat{\eta}$, after which an equation in $\hat{F}_1$ and $\hat{\eta}$ identical in form to (4a) is obtained. Using appropriately rescaled parameters $V_1 = \sqrt{1-C_1/2}\ \hat{V}_1$ and $\hat{\beta} = \left(-\hat{V}_1 + \sqrt{\hat{V}_1^2+4}\right)/2$, a solution for F_1 can be found, namely

$$F_1 = \sqrt{1-C_1/2}\ \hat{\beta}\,(1-\hat{\omega}), \quad \text{where} \quad \hat{\omega} = \frac{\exp(-\hat{\beta}\hat{\eta})}{\hat{\beta}^2} \tag{11}$$

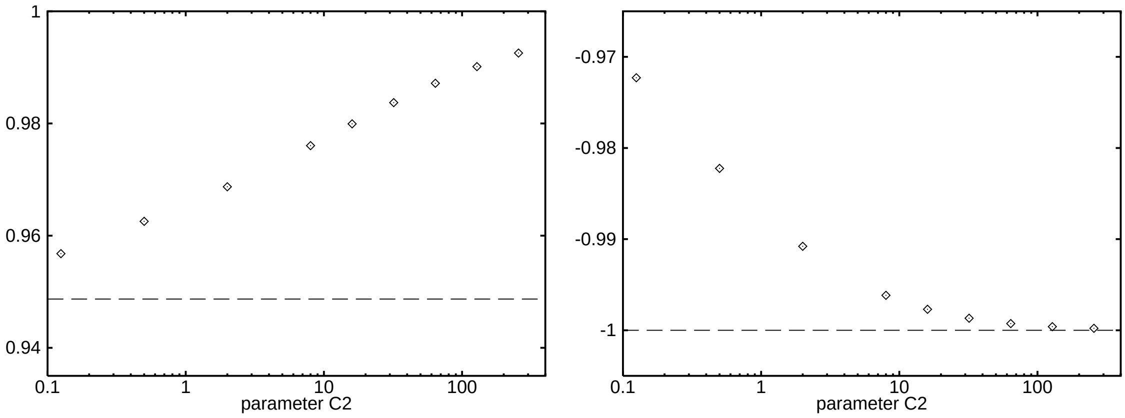
The above solution for $C_2 << 1$ is of the same form as the solution for f_0 given in (8). However, the far field values are different: equation (8) returns $F_1(\infty) = \beta$, while (11) returns the value $F_1(\infty) = \sqrt{1-C_1/2}\ \hat{\beta}$.

To check the validity of the above analysis, a number of computations using a shooting method with fourth order Runge-Kutta, and a quasilinearisation scheme were performed.

Sample results are illustrated in Fig. 1 for the case $C_1 = 0.2$ and $V_1 = 0$ (ie. $\beta = \hat{\beta} = 1$), using the values of $C_2 = 1/8, 1/2, 2, 8, 16, 32, 64, 128$ and 256.

In Fig. 1(a), the computed value of $F_1(\infty)$ is plotted along with the reference value $F_1(\infty) = \sqrt{1 - 0.2/2}$. The illustrated behaviour indicates that $F_1(\infty)$ tends towards the reference value when C_2 is small, consistent with (11), and tends towards the Newtonian outer flow value of $\beta = 1$ as C_2 increases.

In Fig. 1(b), the computed wall value $F_1''(0)$ is shown. The illustrated behaviour confirms that the computed values approach the Newtonian flow value $f_0''(0) = -1$ as C_2 increases.



(a) $F_1(\infty)$ (b) $F_1''(0)$

Figure 1: Values of F_1 and F_1'' for $V_1 = 0$, $C_1 = 0.2$ and $C_2 = 1/8 \to 256$:
(a) $F_1(\infty)$ ($\diamond$) cf. reference value $F_1 = \sqrt{0.9}$ (– –);
(b) $F_1''(0)$ ($\diamond$) cf. reference value $f_0''(0) = -1$ (– –).

3.3 First Order Solution for F_1

We now return to the analysis for $\epsilon = C_1$ and consider (4b). This equation is linear with variable coefficients, and a solution can be expressed as $f_1 = h_c + p_1$, where h_c is a complementary function (CF) and p_1 the particular integral (PI). The CF is obtained via solution of the homogeneous part of (4b), which may be written as

$$h_c''' + \beta h_c'' - \beta\omega(h_c'' + 2\beta h_c' + \beta^2 h_c) = 0 \tag{12}$$

The solution of (12) can be expressed as a linear combination of three independent solutions as $h_c = a_1 h_{c1} + a_2 h_{c2} + a_3 h_{c3}$, where a_1, a_2 and a_3 are constants. To solve this equation, we first note a significant simplification. A change of variable to

$$y = h_c' + \beta h_c \tag{13a}$$

will reduce the order of (12) to the second order equation

$$y'' - \beta\omega(y' + \beta y) = 0 \tag{13b}$$

Now one of the independent functions (h_{c1}, say) is lost in the reduction of order. However, it will correspond to the trivial solution $y = 0$ of (13b), and so can be recovered by solving $h'_{c1} + \beta h_{c1} = 0$. With suitable choices for the arbitrary constants, we obtain

$$h_{c1} = \omega \tag{14}$$

In addition, two non-trivial independent solutions y_1 and y_2 of (13b) are required. One of these can be found by inspection as $y_1 = 1 + \omega$. The second solution y_2 is found most conveniently by first changing variables from η to ω, whereupon (13a) and (13b) become

$$y = -\beta\omega \frac{dh_c}{d\omega} + \beta h_c \tag{15a}$$

$$\omega \frac{d^2y}{d\omega^2} + (1+\omega)\frac{dy}{d\omega} - y = 0 \tag{15b}$$

Using y_1 and classical reduction of order formulae we obtain a solution for y_2 as

$$y_2(\omega) = \exp(-\omega) - (\omega + 1)\int_{\omega} \frac{e^{-v}}{v} dv \tag{16}$$

Finally, h_{c2} and h_{c3} can be obtained by substituting the solutions for y_1 and y_2 above into (15a) and solving. However, before doing so we note that the latter are general solutions to (15b). Thus, an arbitrary scalar multiple of either y_1 or y_2 could be used when solving (15a) for h_{c2} or h_{c3} in order to simplify the final expressions. With this in mind we obtain, with suitable choices for the arbitrary constants involved and lengthy but routine working,

$$h_{c2} = 1 + \beta\eta\omega \quad \text{and} \quad h_{c3} = -2\exp(-\omega) + \mathrm{Ei}(\omega) + 2\omega \mathrm{Ei}(\omega) + \omega \int_{\omega} \frac{\mathrm{Ei}(u)}{u} du \tag{17}$$

where $\mathrm{Ei}(x)$ is the exponential integral function.

In order to introduce algebraic simplifications, and without loss of generality, all upper integration limits were set to $1/\beta^2$. With this modified limit we have, for example, that

$$\mathrm{Ei}(x) = \int_x^{1/\beta^2} \frac{e^{-t}}{t} dt \quad \text{and} \quad \lim_{\eta\to 0} \mathrm{Ei}(\omega) = \lim_{\omega \to 1/\beta^2} \mathrm{Ei}(\omega) = 0 \tag{18}$$

To complete the solution for f_1, it remains to find a PI to (4b) and the values of the constants a_1, a_2 and a_3. To illustrate the analysis, we discuss the solution for $\gamma = 1$, which corresponds to the case of $V_1 = 0$ and $C_2 = 2$ considered in the earlier studies of Hady [4] and Hassanien & Gorla [5]. The case $\gamma \neq 1$ is more complex, and will not be discussed here.

For $\gamma = 1$, (9) is substituted into (4b) and a PI is sought. The latter is found without difficulty by considering a trial solution of the form $p_1 = c_0 + c_1\eta$, which yields

$$p_1 = \frac{3\beta}{4} - \frac{\beta^2}{4}\eta \tag{19}$$

Finally, with $a_0 = 1 - \exp(1/\beta^2)$, the required values of the constants are found to be

$$a_1 = -\frac{\beta^3}{4}\left(a_0 + \frac{1+a_0}{1+\beta^2}\right), \quad a_2 = -\frac{\beta^3}{4}\frac{1+a_0}{1+\beta^2} \quad \text{and} \quad a_3 = \frac{\beta}{4} \tag{20}$$

To check the present analysis, we compared our results with earlier studies and our own computations, and some key results are reported in Table 1. Our computed and analytic results for the surface conditions $F_1''(0)$ and $F_2'(0)$ are both in excellent agreement with

	Computed (This work)	Analytic (This work)	Computed (Ref. [5])	Analytic (Ref. [4])
$-F_1''(0)$	0.99078	0.99080	0.99081	0.97500
$F_2'(0)$	0.25120	0.25000	0.25121	0.25316

Table 1: Values of $F_1''(0)$ and $F_2'(0)$ for $\beta = 1$, $C_1 = 0.2$ and $C_2 = 2$.

the numerical solution of Hassanien & Gorla [5]. By comparison, Hady's [4] approximate analytic solution is significantly worse.

Finally, in Table 2, computed and perturbation solutions are compared for different values of $C_1 \to 1$. The tabulated data shows that the approximation $F_1 \approx f_0 + C_1 f_1$ provides an accurate closed form solution for the streamfunction over a range of C_1 values. Evidently, the contribution from the higher order functions (f_n, $n > 1$) to (3) is significantly smaller than f_1.

C_1	0.1	0.3	0.5	0.7	0.9
$-F_1''(0)$ (Computed)	0.99540	0.98617	0.97691	0.96766	0.95845
$-F_1''(0)$ (Analytic)	0.99540	0.98620	0.97701	0.96781	0.95861

Table 2: Analytic and computed $F_1''(0)$ values for $\beta = 1$, $C_2 = 2$ and different C_1.

In conclusion, the perturbation analysis described above yields results which are in good agreement with supporting computations and existing studies, and also provides a vehicle for analysing certain limiting solutions to the governing equations for micropolar flow over a stretching sheet. In work currently underway, we are investigating the generality of the present approach to describe the flow for even greater ranges of the model parameters involved.

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